Los A.V.

# Numbers and bases

## The concept behind a number

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This booklet contains content from the book *2 to 16, for young programmers, 1984, USSR.*

Ч.Н. Ролич (1981). *от 2 до 16*. Min: Минск. p. 3-5.

## Introduction

Please take a look at these expressions:

$$2 + 2 = 11$$

$$5 \times 5 = 31$$

$$10 - 1 = 1$$

$$\frac{53}{6} = 8$$

$$9 + 1 = A$$

Probably, they look a bit strange, but, in fact, they are all true. That is because we are dealing with different numerical systems, also called number bases. As you might have figured out, the first expression is written in base 3, the second in base 8, the third in base 2 (also: binary), the fourth in base 9, and the final expression is base 16 (also: hexadecimal).

The concept of numerical systems is as old as our civilisation. The old Greeks had a numerical system, already in the antiquity. Although we are not using it to perform any mathematical calculations, we still use it to, for instance, write kings' name, e.g. *Charles XII*. Imagine how difficult it would have been, if we continued to use the old numerical system today[1].

However, nowadays, people still have a use of other numerical system, such as the binary (base 2), octal (base 8), hexadecimal (base 16). You have probably heard that these systems are mainly used in the computer's world, and certainly, because of a good reason. Binary numbers are great because it is easy to implement these in an electric circuit.*1* stands for *on*, and *0* stand for *off*. Morse code uses binary numbers to express each letter in the English alphabet. Unfortunately, because it only contains two symbols, numbers expressed in binary get really big, and in a long run, it makes calculation really difficult, and inefficient.

$$587_{10} = 1001001011_2$$

As you can see the tree digit number in base 10 turns into a ten digit long binary number. It works, but why?

[1]Please note that the system used by the Greeks differs from base 2, base 5, and our system. Even though the principle is similar, the way it expresses a number is entirely different.

One of the theories of how a numerical system was created is the *finger counting*. The most common system used in everyday life is base 10, but there are also base 5, base 20 as well. All of these are based on the way we usually count – using fingers.

We begin on 1 and continue till 10, and then we restart. However, in some cultural groups, base 20 is being used. That means that it starts from 1and goes up to 20, until it restarts again. You can probably guess – this system is not only using fingers, but also toes. Let us investigate base 20!

## Investigating base 20

In order to keep it simple, let's decide our digits in our new system using letters from the English alphabet.

| Base 20 | Base 10 |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | 6 |
| 7 | 7 |
| 8 | 8 |
| 9 | 9 |
| A | 10 |
| B | 11 |
| C | 12 |
| D | 13 |
| E | 14 |
| F | 15 |
| G | 16 |
| H | 17 |
| I | 18 |
| J | 19 |

Base 10 is the system we are used to, but base 20 is our invented system.

Probably, you would ask, why there is no symbol for *20*. The explanation to that is simple; think of it as it is in base 10. The number *10* is actually built up of two digits – namely *1* and *0*. Our system consists of only 10 symbols:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

So, the question is, how would 20 be expressed, if not using one single symbol? The number 20 in base 20, the number 10 in base 10 are basically indicating that we should restart our counting, by adding a zero at the end. Therefore, 20 in base 20 is:

$$10_{20} = 20_{10}$$

So, if we would place base 20 numbers in order, we would get following (from 0 two 21):

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, G, H, I, J, 10, 11$$

The same in our system (base 10) it would look like:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21$$

You might already see, using base 20, we save space, i.e. less symbols required to express a number.

## What is a number

In order to understand a numerical system even better, you need to know how a number in our system might be expressed.

Say we take *2012* as our number. Hopefully, you will agree that it is the same as:

$$2\ 0\ 1\ 2$$

$$2 \times 1000 + 0 \times 100 + 1 \times 10 + 2 \times 1$$

| | |
|---|---|
| $10^0$ | Digits |
| $10^1$ | Tens |
| $10^2$ | Hundreds |
| $10^3$ | Thousands |

As you can see, as we go to the left, i.e. from tens to hundreds, etc, we increase the power of 10 (which is the base) with 1.

By understanding that, let's look at why 10 in base 20, is 20 in our system.

$$10$$

$$1 \times 20^1 + 0 \times 20^0$$

Note, the same rule applies to any base. In base 5 for instance, you would have fives, fifth tens, five hundreds, etc.

| | |
|---|---|
| $20^0$ | Digits |
| $20^1$ | Twenties |
| $20^2$ | Two hundreds |
| $20^3$ | Two thousands |

In the same way, try to predict what values $A2$, $JA$, $100$ have!

## From a base to another

As it is described in the previous section, we can express any number in any numerical systems, in base 10 (i.e. our system). However, if we would like to express a number that is in base 10 originally, in any other numerical system, we will need to use division, using a remainder.

Because in discrete mathematics, we appreciate whole numbers, in this division method[2], we are going to get a quotient and a remainder. Let's take 20/3 as an example.

1. First, count the amount of times 3 goes in 20. That's 6.
2. Multiply 3(denominator) by 6(no. of times 3 goes in 20) to get 18.
3. Now, the difference between 20 and 18 is our remainder, which is 2.

[2] This is the same as *modulo arithmetic*.

In terms of fractions, we have:

$$\frac{20}{3} = 6 + \frac{2}{3}$$

Now we are able to rewrite any number in any base to the base we want. Let's convert 17 in base 10, to base 3.

| Number | Quotient | Remainder |
|---|---|---|
| 17 | | 1 |
| | | 2 |
| | | 2 |